Gauss Divergence Theorem

$$\int_S u \cdot \vec{n} \, dS = \int_V \nabla \cdot u \, dV$$

index notation

$$\int_S u_i n_i \, dS = \int_V \partial_i (u_i) \, dV$$

Reynold's Transport Theorem (Webpage)

$$\frac{d}{dt} \int_{V(t)} f \, dV = \int_{V(t)} \partial_t f + \nabla \cdot (f u) \, dV$$

$$= \int_{V(t)} \partial_t f + \partial_i (f u_i) \, dV$$

Mass Conservation

$$\frac{d}{dt} \int_{V(t)} \rho \, dV = \int_{V(t)} \partial_t \rho + \nabla \cdot (\rho u) \, dV = 0$$

$$\partial_t \rho + \nabla \cdot (\rho u) = 0$$

Conservation of Momentum

$$\frac{d}{dt}\int_{V(t)} \rho u \, dV = \int_{V(t)} \partial_t(\rho u) + \partial_i(\rho u^i u^j)\, dV$$

$$= \int_{V(t)} \partial_t(\rho u) + \nabla\cdot(\rho u \otimes u)\, dV$$

$$u^i u_i = u \cdot u \in \mathbb{R}^3$$

$$u^i u^j = \begin{bmatrix} u^1u^1 & u^1u^2 & u^1u^3 \\ \vdots & \vdots & \vdots \\ u^3u^1 & u^3u^2 & u^3u^3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$= u \otimes u$$

Body force $\int_{V(t)} \rho\, b\, dV \qquad b \in \mathbb{R}^3$

shear force $-pI\vec{n} + \tau\cdot\vec{n}$

(τ: linear transformation)

$$\sigma = -pI + \tau \qquad \sigma_{ij} = \sigma_{ji}$$

For Newtonian Fluid

$$\sigma = -pI + 2\mu D$$

(D: deformation tensor)

$$\int_{S(t)} \sigma \cdot n \, dS = \int_{S(t)} \sigma_{ij} n^j \, dS$$

$$= \int_{V(t)} \partial_j (\sigma^{ij}) \, dV$$

$$= \int_{V(t)} \nabla \cdot \sigma \, dV$$

$$\nabla \cdot \sigma = \nabla \cdot (-p\bar{I} + \tau) = -\nabla p + \nabla \cdot \tau$$

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) = -\nabla p + \nabla \cdot \tau + \rho b$$

Energy Conservation

E – energy per unit mass

$$\frac{d}{dt} \int_{V(t)} \rho E \, dV = \int_{V(t)} \partial_t(\rho E) + \nabla \cdot (\rho E u) \, dV$$

body force rate $\int_{V(t)} \rho b \cdot u \, dV$

shear force rate

$$\int_{S(t)} (\sigma \cdot n) \cdot u \, dS$$

$$= \int_{S(t)} \sigma^{ij} n_j u_i \, dS$$

$$= \int_{V(t)} \partial_j (\sigma^{ij} u_i) \, dV$$

$$= \int_{V(t)} \nabla \cdot (\sigma \cdot u) \, dV$$

$$\nabla \cdot (\sigma \cdot u) = -\nabla \cdot (pu) + \nabla \cdot (\tau \cdot u)$$

body heat source

$$\int_{V(t)} \rho r \, dV$$

heat flux

$$-\int_{S(t)} q \cdot \vec{n}\, dS = -\int_{V(t)} \nabla \cdot q\, dV$$

$$\partial_t(\rho E) + \nabla \cdot \left((\rho E + p)\, u \right)$$

$$= \nabla \cdot (\tau \cdot u) - \nabla \cdot q + \rho b \cdot u + \rho r$$