Gauss Divergence Theorem

$$\int_S u \cdot \vec{n} \, dS = \int_V \nabla \cdot u \, dV$$

index notation

$$\int_S u_i n_i \, dS = \int_V \partial_i (u_i) \, dV$$

Reynold's Transport Theorem (Webpage)

$$\frac{d}{dt} \int_{V(t)} f \, dV = \int_{V(t)} \partial_t f + \nabla \cdot (f u) \, dV$$

$$= \int_{V(t)} \partial_t f + \partial_i (f u_i) \, dV$$

Mass Conservation

$$\frac{d}{dt} \int_{V(t)} \rho \, dV = \int_{V(t)} \partial_t \rho + \nabla \cdot (\rho u) \, dV = 0$$

$$\partial_t \rho + \nabla \cdot (\rho u) = 0$$

C.P. corresponds to "Covariant Physics".

Define

$$F(t) = \int_{V(t)} f(x,t)\,\mathrm{d}V,$$

where $V(t)$ always consists of the same fluid particles. We assume the flow map $\mathcal{X}$ is smooth and invertible, f is smooth. Spatial coordinates & metric are time-independent.

Spatial coordinates x^i & material coordinates X^A

$$i, j, k \qquad A, B, C$$

Spatial metric g_{ij} & material metric G_{ij}

Denote $\mathrm{d}^3x = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$.

By Eq (2.137) in C.P.

$$\mathrm{d}V = \sqrt{|g|}\,\mathrm{d}^3x = \mathcal{J}\,\mathrm{d}^3x$$

Let V_0 denote a fixed ref. configuration of the fluid. Each fluid particle is assigned a fixed material label X^A.

Therefore,

$$\frac{\mathrm{d}X^A}{\mathrm{d}t} = 0$$

Current spatial coordinates

$$x^i = \mathcal{X}^i(X^A, t)$$

Exercise 1. Show that for a given particle

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} = \left.\frac{\partial \mathcal{X}^i}{\partial t}\right|_X$$

Let

$$\Lambda^i{}_A = \frac{\partial x^i}{\partial X^A} = \frac{\partial \mathcal{X}^i}{\partial X^A}$$
$$\Lambda^A{}_i = \frac{\partial X^A}{\partial x^i} = \frac{\partial X^A}{\partial \mathcal{X}^i}.$$

We know that

$$\Lambda^A{}_i \Lambda^i{}_B = \delta^A{}_B$$
$$\Lambda^i{}_A \Lambda^A{}_j = \delta^i{}_j.$$

By Eq (2.127) in C.P., $\mathcal{J} = |\Lambda|$

$$\mathrm{d}V = \mathrm{d}^3x = \mathcal{J}\,\mathrm{d}^3X$$

Therefore,

$$F(t) = \int_{V_0} f(\mathcal{X}(X,t),t)\mathcal{J}(X,t)\,\mathrm{d}^3X$$

$$\begin{aligned}\frac{\mathrm{d}F(t)}{\mathrm{d}t} &= \int_{V_0} \frac{\partial}{\partial t}\left[f(\mathcal{X}(X,t),t)\mathcal{J}(X,t)\right]\mathrm{d}^3X \\ &= \int_{V_0}\left[\mathcal{J}\left.\frac{\partial f(\mathcal{X}(X,t),t)}{\partial t}\right|_X + f\frac{\partial \mathcal{J}}{\partial t}\right]\mathrm{d}^3X\end{aligned} \tag{17}$$

First term

$$\begin{aligned}\left.\frac{\partial f(\mathcal{X}(X,t),t)}{\partial t}\right|_X &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\left.\frac{\partial x^i}{\partial t}\right|_X \\ &= \partial_t f + u^i \partial_i f \\ &= \frac{\mathrm{D}f}{\mathrm{D}t}\end{aligned}$$

Second term $\mathcal{J} = \det[\Lambda^i{}_A]$

$$\begin{aligned}\frac{\partial \Lambda^i{}_A}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial x^i}{\partial X^A} = \frac{\partial}{\partial X^A}\frac{\partial x^i}{\partial t} = \frac{\partial}{\partial X^A}u^i(\mathcal{X}(X,t),t) \\ &= \frac{\partial u^i}{\partial x^j}\frac{\partial x^j}{\partial X^A} = \partial_j u^i \Lambda^j{}_A\end{aligned}$$

By Eq (8.108) in C.P.

$$\mathrm{d}|M| = |M|\,\mathrm{Tr}(M^{-1}\mathrm{d}M)$$

we have

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial t} &= \mathcal{J}\,\mathrm{Tr}\left(\Lambda^{-1}\frac{\partial \Lambda}{\partial t}\right)\\ &= \mathcal{J}\Lambda^{A}{}_{i}\frac{\partial \Lambda^{i}{}_{A}}{\partial t}\\ &= \mathcal{J}\Lambda^{A}{}_{i}(\partial_j u^i)\Lambda^{j}{}_{A}\\ &= \mathcal{J}\partial_j u^i \delta^{j}{}_{i} = \mathcal{J}\partial_i u^i\end{aligned}$$

i.e.

$$\frac{1}{\mathcal{J}}\frac{\partial \mathcal{J}}{\partial t} = \nabla\cdot\mathbf{u}$$

$$\begin{aligned}\frac{\mathrm{d}F(t)}{\mathrm{d}t} &= \int_{V_0}\frac{\partial}{\partial t}\left[f(\mathcal{X}(X,t),t)\mathcal{J}(X,t)\right]\mathrm{d}^3X\\ &= \int_{V_0}\left[\mathcal{J}\left.\frac{\partial f(\mathcal{X}(X,t),t)}{\partial t}\right|_X + f\frac{\partial \mathcal{J}}{\partial t}\right]\mathrm{d}^3X\\ &= \int_{V_0}\left[\mathcal{J}(\partial_t f + u^i\partial_i f) + f\mathcal{J}\partial_i u^i\right]\mathrm{d}^3X\\ &= \int_{V_0}\left(\partial_t f + u^i\partial_i f + f\partial_i u^i\right)\mathrm{d}V\\ &= \int_{V_0}\left(\frac{\mathrm{D}f}{\mathrm{D}t} + f\partial_i u^i\right)\mathrm{d}V\\ &= \int_{V_0}\left(\partial_t f + \underbrace{\partial_i(fu^i)}_{\nabla\cdot(f\mathbf{u})}\right)\mathrm{d}V\end{aligned}$$

Conservation of Momentum

$$\frac{d}{dt}\int_{V(t)} \rho u \, dV = \int_{V(t)} \partial_t(\rho u) + \partial_i(\rho u^i u^j)\, dV$$

$$= \int_{V(t)} \partial_t(\rho u) + \nabla\cdot(\rho u \otimes u)\, dV$$

$$u^i u_i = u \cdot u \in \mathbb{R}^3$$

$$u^i u^j = \begin{bmatrix} u^1u^1 & u^1u^2 & u^1u^3 \\ \vdots & \vdots & \vdots \\ u^3u^1 & u^3u^2 & u^3u^3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$= u \otimes u$$

Body force $\int_{V(t)} \rho\, b\, dV \qquad b \in \mathbb{R}^3$

shear force $-pI\vec{n} + \tau\cdot\vec{n}$

(τ: linear transformation)

$$\sigma = -pI + \tau \qquad \sigma_{ij} = \sigma_{ji}$$

For Newtonian Fluid

$$\sigma = -pI + 2\mu D$$

(D: deformation tensor)

$$\int_{S(t)} \sigma \cdot n \, dS = \int_{S(t)} \sigma_{ij} n^j \, dS$$

$$= \int_{V(t)} \partial_j (\sigma^{ij}) \, dV$$

$$= \int_{V(t)} \nabla \cdot \sigma \, dV$$

$$\nabla \cdot \sigma = \nabla \cdot (-p\bar{I} + \tau) = -\nabla p + \nabla \cdot \tau$$

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) = -\nabla p + \nabla \cdot \tau + \rho b$$

Energy Conservation

E – energy per unit mass

$$\frac{d}{dt} \int_{V(t)} \rho E \, dV = \int_{V(t)} \partial_t(\rho E) + \nabla \cdot (\rho E u) \, dV$$

body force rate $\int_{V(t)} \rho b \cdot u \, dV$

shear force rate

$$\int_{S(t)} (\sigma \cdot n) \cdot u \, dS$$

$$= \int_{S(t)} \sigma^{ij} n_j u_i \, dS$$

$$= \int_{V(t)} \partial_j (\sigma^{ij} u_i) \, dV$$

$$= \int_{V(t)} \nabla \cdot (\sigma \cdot u) \, dV$$

$$\nabla \cdot (\sigma \cdot u) = -\nabla \cdot (pu) + \nabla \cdot (\tau \cdot u)$$

body heat source

$$\int_{V(t)} \rho r \, dV$$

heat flux

$$-\int_{S(t)} q \cdot \vec{n}\, dS = -\int_{V(t)} \nabla \cdot q\, dV$$

$$\partial_t(\rho E) + \nabla \cdot \left((\rho E + p)\, u \right)$$

$$= \nabla \cdot (\tau \cdot u) - \nabla \cdot q + \rho b \cdot u + \rho r$$