Helmholtz Decomposition

$$u = u_0 + \frac{1}{2}(\nabla \times u) \times \delta x + \nabla \phi \qquad \phi = \delta x^T S \delta x$$

u_0: transition; $\frac{1}{2}(\nabla \times u) \times \delta x$: rotation; $\nabla \phi$: deformation

Notation

$$x \in \mathbb{R}^3 \qquad x = \sum_{i=1}^{3} x^i e_i \equiv x^i e_i$$

$$u \in \mathbb{R}^3 \qquad u = \sum_{i=1}^{3} u^i e_i \equiv u^i e_i$$

$$\nabla f \overset{\text{NP}}{=} \begin{bmatrix} \frac{\partial f}{\partial x^1} \\ \frac{\partial f}{\partial x^2} \\ \frac{\partial f}{\partial x^3} \end{bmatrix} \Rightarrow \frac{\partial}{\partial x^i} = \partial_i$$

Kronecker Delta $$\delta^{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Levi-Civita Symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{if } \epsilon_{iik}, \epsilon_{ijj}, \cdots \end{cases}$$

Lemma ("Covariant Physics" (1.107))

$$\epsilon_{ijk}\epsilon^{mnk} = (\delta^m_i \delta^n_j - \delta^n_i \delta^m_j)$$

Proposition $u, v \in \mathbb{R}^3$ $\quad (u \times v)^i = \epsilon^i_{jk} u^j v^k$

$$u^i = u^i(0) + \sum_{j=1}^{3} \frac{\partial u^i}{\partial x^j} \delta x^j$$

Without loss of generality, let $u^i(0) = 0$

$$u^i = \frac{\partial u^i}{\partial x^j} \delta x^j = (\partial_j u^i)\, \delta x^j$$

Let $T_{ij} = T^i_j \triangleq \partial_j u^i \qquad u^i = T_{ij}\, \delta x^j$

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{pmatrix} \delta x^1 \\ \delta x^2 \end{pmatrix} = \begin{pmatrix} T_{11}\delta x^1 + T_{12}\delta x^2 \\ T_{21}\delta x^1 + T_{22}\delta x^2 \end{pmatrix} = \begin{pmatrix} T_{1j}\delta x^j \\ T_{2j}\delta x^j \end{pmatrix}$$

$$\underbrace{T \cdot \delta x}$$

Def. (Symmetric tensor) $S_{ij} = S_{ji}$

Def. (Anti-Symm tensor) $A_{ij} = -A_{ji}$

Theorem $T_{ij} = A_{ij} + S_{ij}$

votícity ten.

$$A_{ij} = \frac{1}{2}(T_{ij} - T_{ji})$$

$$S_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$$

deformation tensor

$$u_i = T_{ij}\,\delta x^j = A_{ij}\,\delta x^j + S_{ij}\,\delta x^j$$

Anti-symm term / Voticity tensor

Define $\omega^j \equiv -\frac{1}{2}\epsilon^{jmn} A_{mn}$

Prop. $A_{ij}\, x^i = \epsilon_{ijk}\, w^j x^k$

$$Ax = w \times x$$

proof:

$$\epsilon_{ijk} w^j x^k = -\frac{1}{2}\epsilon_{ijk}\epsilon^{jmn} A_{mn} x^k$$

$$= \frac{1}{2}\epsilon_{ikj}\epsilon^{mnj} A_{mn} x^k$$

$$= \frac{1}{2}\left(\delta_i^m \delta_k^n - \delta_i^n \delta_k^m\right) A_{mn} x^k$$

$$= \frac{1}{2}\left(A_{ik} - A_{ki}\right) x^k$$

$$= A_{ik} x^k = A_{ij} x^j$$

Prop. $w^i = \frac{1}{2}\epsilon^i{}_{jk}\, \partial_j u^k$

$$w = \frac{1}{2}\nabla \times u$$

proof:

$$w^i = -\frac{1}{2}\epsilon^{imn} A_{mn}$$

$$= -\frac{1}{2}\epsilon^{imn}\frac{1}{2}\left(T_{mn} - T_{nm}\right)$$

$$= \frac{1}{4}\epsilon^{imn}\left(T_{nm} - T_{mn}\right)$$

$$= \frac{1}{4}\epsilon^{imn}\left(\partial_m u^n - \partial_n u^m\right)$$

$$= \frac{1}{4}\epsilon^{imn}\partial_m u^n + \frac{1}{4}\epsilon^{imn}\partial_m u^n$$

$$= \frac{1}{2}\epsilon^{imn}\partial_m u^n$$

$$A\delta x = \frac{1}{2}(\nabla \times u) \times \delta x$$

Symmetric term

$$Sx = ?$$

$$\phi = \frac{1}{2}x^T S x \qquad \nabla\phi = Sx$$

$$Sx = \nabla\phi \qquad \phi : \mathbb{R}^3 \to \mathbb{R}$$

HW Define $\phi = \frac{1}{2}x^i S_{ij} x^j$

Prove

$$\partial_i \phi = S_{ij} x^j$$